How Much Does That House Really Cost?

Report

Vaami Satrasala and Benjamin Jardine

1 Introduction

We have two close friends who require assistance purchasing a house in Boulder; unfortunately, they seem to have forgotten their differential equation skills and have decided to requisition the aid of us engineering students. To assist them, we have put together the following report regarding the analysis of different kinds of mortgage structures, including fixed and adjustable rates, different monthly payment values, and long-term vs short-term payment plans. Our friends seem inclined to borrow $750,000, so that is the initial loan value we will default for all our analysis.

To obtain our results, we used MATLAB and PyCharm. We used MATLAB to create the various plots by inputting the true solution to the loan value functions, as well as by plotting Eulerian approximations by iterating through a vector of time values. We used PyCharm to perform Eulerian approximations to determine the time at which loans were paid off with both fixed and adjustable-rate mortgages, as well as to calculate the value of the loan after 5 years for different yearly compounding amounts given a fixed interest rate.

2 Analysis of Fixed Rate Mortgages

Our friends want us, first, to analyze a fixed rate mortgage. The initial loan they want to take is $750,000, and the fixed rate of interest that we will analyze for the first possible payment scenario is set at 3%. The value of the loan over time is governed by the function$A(t)$:

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{(nt)} \qquad (1a)$$

Where $A(0) = A_0$ is the initial value of the loan, r is the interest rate, and n is the number of times the loan is compounded per year. For our purposes the equation looks like:

$$A(t) = (7.5 * 10\hat{}5)\left(1 + \frac{0.03}{n}\right)^{(nt)} \qquad (1b)$$

If the loan is compounded continuously, we take the $\lim_{n\to\infty} A(t)$. The resulting function is Equation (1c):

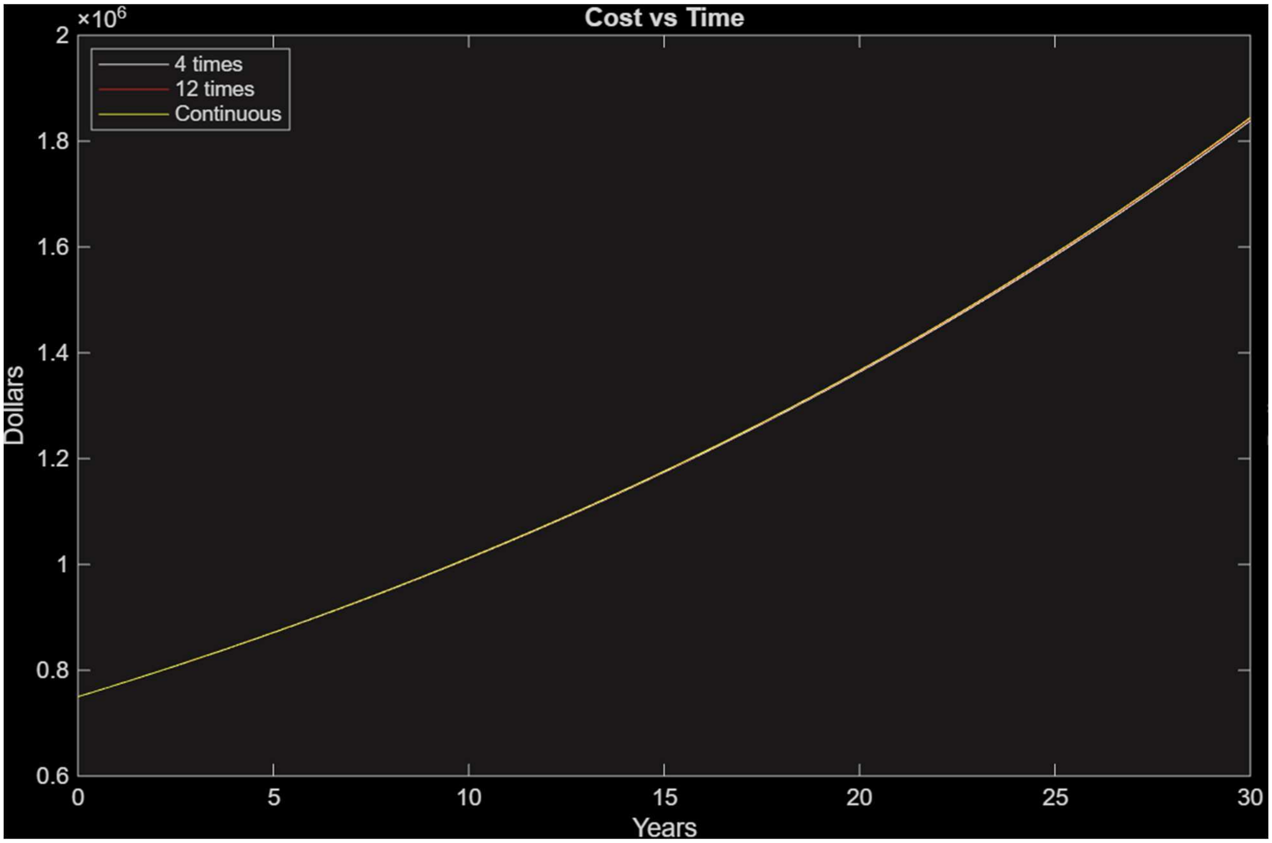
$$A(t) = 7.5 \cdot 10^5 e^{(0.3t)} \qquad (1c)$$

To gain a general understanding of the effects of various compounding frequencies, we took the total value of the loan with yearly compounding amounts of once, twice, four times, twelve times, and continuously over the course of 5 years. The end values are compared in the table below:

Table 1 - Amount of the Loan over Five Years

Compounding periods per year {n}:	**Loan Amount {A(t)}**
1	\$869455.56
2	\$870405.62
4	\$870888.11
12	\$871212.59
$\lim_{n\to\infty}\left(A(t)\right)$	\$871375.68

We note that as the yearly compounding amount increases, so does the loan amount. If we plot the equations governing the last three rows of Table 1 on a time plot from $t = 0$ to $t = 30$, where t is in years, we get the following:



Relatively speaking, the results over 30 years are very similar, and are almost indistinguishable in this graph. Still, as the amount of annual compounding increases, so does the loan value. For the purposes of our analysis, we shall simply use the continuous compounding function, as it is the most expensive option, and using it will give our friends an understanding of the worst-case payment scenario.

In the case that our friends wish to be able to eventually pay off the loan rather than remain in increasing debt forever, they should make a monthly payment, *p,* in dollars. In this case, the model of the differential equation for the loan value becomes:

$$A'(t) = rA(t) - 12p \qquad (2a)$$

If we set the rate of change of the total loan amount, $A'(t)$, equal to zero to find the 'equilibrium solutions,' we find that $A'(t) = 0$ when $A(t) \ = \frac{12p}{r}$. This means that if our initial loan value $A_0 = \frac{12r}{p}$, or, alternatively, $p = \frac{rA(t)}{12}$for some monthly payment *p*, the rate of change of the loan forever remains 0, and our friends' total debt will never change as the rate of increase of the loan will perfectly negate the monthly payments made. They will always owe the initial amount to the bank.

If $A_0 > \frac{12r}{p}$, over time your total debt will increase, since $A'(t) = rA(t) - 12p$ at any given time, so if the loan value at time t = 0 is$A_0 > \frac{12r}{p}$(or, alternatively, if $p \geq \frac{rA}{12}$for a fixed initial value), the rate of change will be positive. This will increase the loan value to a greater value as time increases in the positive direction, resulting in another positive derivative at the new time value, and so on. Hence, the loan value increases forever. Using the same logic, we can conclude that if our initial loan value $A_0 < \frac{12r}{p}$or alternatively if $p \geq \frac{rA}{12}$,)$A'(t) < 0$, for any positive time value, and over time the debt will eventually go to zero. This means that the equilibrium solution is unstable, and our friends should begin with a loan value of $A_0 > \frac{12r}{p}$, or, for a given loan value $p \geq \frac{rA}{12}$.

To determine analytically the value of the loan itself:

If $A(0) = A_0$, the interest rate is r, and the constant monthly payment is$p$ (both some arbitrary real constant), and time t is variable, solving equation (2a) looks like:

$$\frac{dA}{dt} = rA - 12p$$

Solve by Separation of Variables:

$$\int \frac{dA}{rA - 12p} = \int dt$$

$$\frac{1}{r} ln|rA - 12p| = t + C$$

$$|rA - 12p| = e^{rt+r}$$

$$|rA - 12p| = e^{rt}e^{C}$$

$$rA - 12p = Ce^{rt}$$

$$A = \frac{1}{r}(Ce^{rt} + 12p)$$

Apply Initial Condition, $A(0) = A_0$:

$$A(0) = A_0 = \frac{1}{r}(Ce^{r*0} + 12p)$$

$$A_0 = \frac{1}{r}(C + 12p)$$

$$rA_0 = C + 12p$$

$$C = rA_0 + 12p$$

Therefore:

$$A(t) = \frac{1}{r}((rA_0 + 12p)e^{rt} + 12p) \qquad \text{(2b)}$$

For our friends $A_0 = \$750{,}000$, so substituting into (2b):

$$A(t) = \frac{1}{r}((750000r + 12p)e^{rt} + 12p) \qquad \text{(2c)}$$

To give our friends an idea of the costs and benefits of taking a longer loan period vs a shorter one, we will determine the monthly payment value necessary to successfully pay off a $750,000 loan over both 10 and 30 years with interest rates of 3% and 5%, respectively. To compare the ideal monthly payment of (x) a 30-year mortgage with an interest rate of 5%, and (y) a 10-year mortgage with an interest rate of 3%, we solve equation (2c) for p(calculation in Appendix C), substitute in our constants, and round up one cent from that amount. We find that the ideal payment for paying off (y) is equal to $7234.31, and the ideal payment for paying off (x) is equal to $4022.56.

Despite the immediate benefits of a lower monthly payment, in the long term that option comes with a hefty price that our friends should be aware of. By summing up the monthly payments over the loan duration, we find that if they take the 30-year loan, they will pay approximately $580,002.45 more in interest than the 10-year loan over the entire course of the loan. They will have paid ~193% of the initial loan value by the end of 30 years, compared to ~115% of the initial value by the end of 10.

Table 2 - Loan Comparison

	10 Years	**30 Years**
Initial Loan:	$750,000.00	$750,000.00
Interest Rate:	0.03%	.05%
Ideal Monthly Payment:	$7234.30	$4022.55
Total Amount Paid:	$868,116.58	$1,448,119.03
Total Interest Paid:	$118,116.58	$698,119.03
Difference in Interest Paid:	$580,002.45	

Since the amount of the loan increases proportionally to itself, our friends should make as large of a down payment as possible, because this drastically decreases the amount paid in interest, and the ideal monthly payment.

For example, if our friend paid $100,000 up front (resulting in an initial loan value of $650,000) Table (2) would instead look like this:

Table 3 - Loan Comparison with a Down Payment

	10 Years	**30 Years**
Initial Loan:	$650,000.00	$650,000.00
Interest Rate:	0.03%	.05%
Ideal Monthly Payment:	$6269.73	$3486.21
Total Amount Paid:	$752,367.7	$1,255,036.49
Total Interest Paid:	$102,367.7	$605,036.49
Difference in Interest Paid:	$502,668.79	
Amount Saved:	$115,748.88	$193,082.54

Our friends would save a total of $115,748.88 over the 10-year loan, and $193,028.54 over the thirty-year loan. Something interesting to note is how in both cases, our friends will still be paying ~193% and ~116% of the initial loan value by the end of the loan, even though they'll end up paying far less by the end of the loan in both cases than they would without a down payment. It's worth it to make a larger down payment in both cases; however, the greater down payment will save you more money over the 30-year loan period than it will the 10-year loan period, so it's an even more beneficial decision to make if our friends plan on paying off their loan over the longer time period.

The benefits of the longer loan are that you will have a lower monthly payment, by a little more than half the ideal payment of the shorter loan. This is a good option if you don't think your future income will be sufficient to comfortably pay the much higher monthly payment. However, longer loan periods usually have a higher interest rate, and the lower ideal payment means that you have more money for the higher interest rate to compound, and you end up paying a little over half a million dollars more for the same house than you would with a shorter loan period.

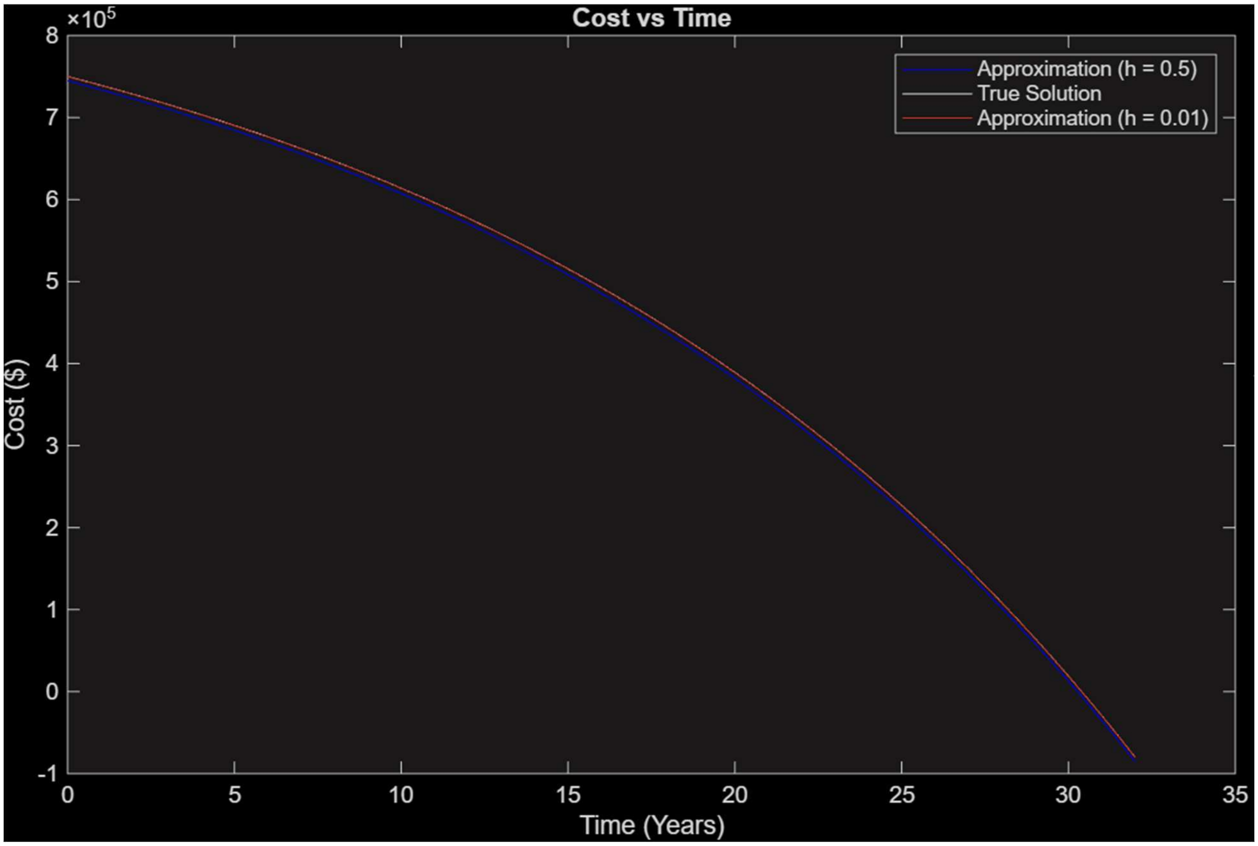
The shorter loan has opposite benefits and detriments to the longer loan. You will have a much higher monthly payment, however, the time that it takes to be paid off is much shorter, the interest rate is lower, and because you are reducing the debt amount by more over time, you will pay of the loan much quicker, and pay far less by the end of the loan.

In both cases, it's very beneficial to make a down payment, so if our friends can afford to do so, we advise that they do, especially if they plan on taking out a longer loan, since that option has an even greater benefit to making a down payment.

Now, rather than solving for monthly payment values given a time frame within which to pay off the loan, we will find the time it takes to pay off the loan given a variable monthly payment that our friends might want to make. This time, we will use Leonard Euler's method to approximate the solution to the differential equation, rather than directly solving for the loan value over time since it would be difficult to find an analytic solution given a variable interest rate. To ensure accuracy, we will first use

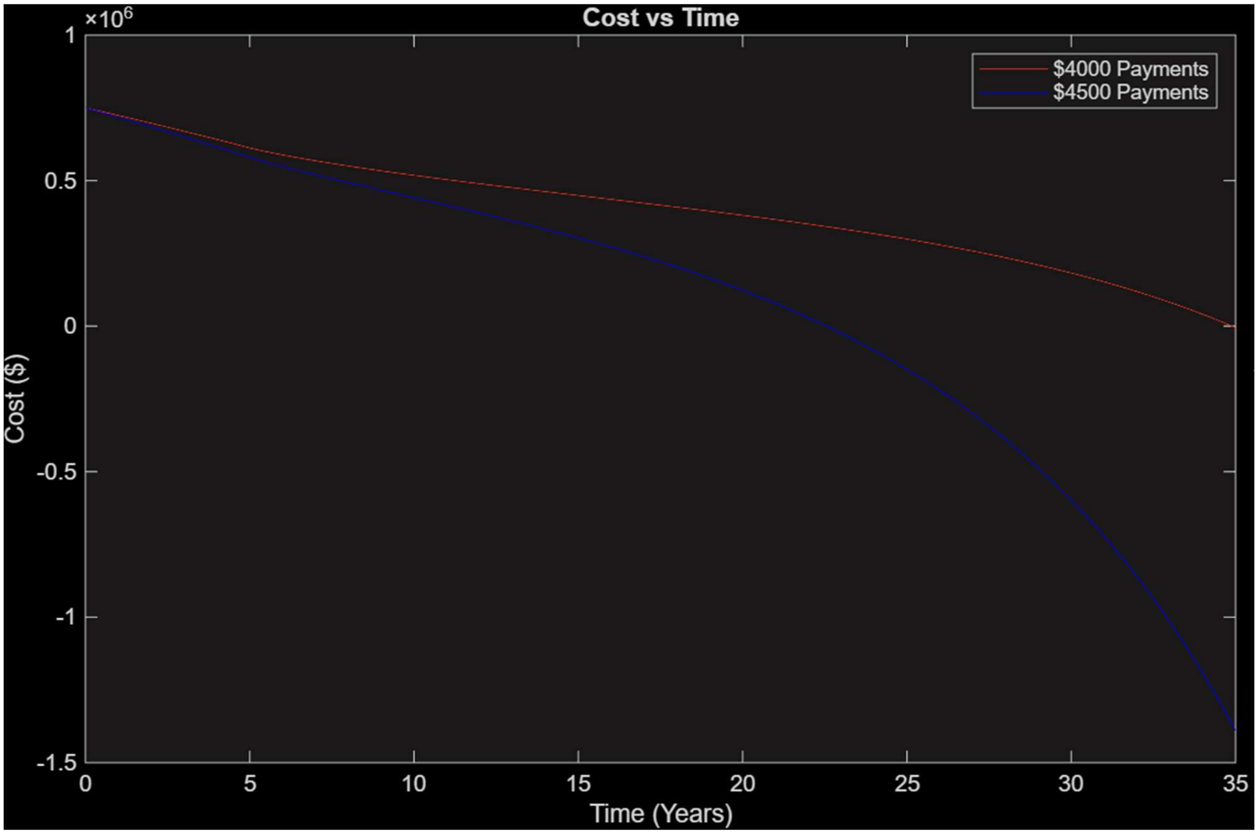
Euler's method to approximate the loan value with a fixed interest rate and ensure it is a good approximation.

We plot the true solution vs an Eulerian approximation of it. For this, we used an initial loan value of \$750,000, a fixed interest rate of 5%, and a monthly payment of \$4000. We used two different step sizes: 0.01 and 0.5.



It's somewhat difficult to see, but the approximation with step size of h = 0.01 very closely approximates the analytic solution of the loan value over the course of 30 years, much more closely than the h = 0.5 approximation. Thus, for the variable rate mortgage approximations, we will use a step size of h = 0.01.

3 Numerical Analysis of Adjustable-Rate Mortgages



If our friends commit to a monthly payment of $4000, it will take approximately 35 years to pay off their loan. On the other hand, if they commit to a slightly higher monthly payment of $4500, they will pay off their loan in approximately 22 years and 8 months.

It is advisable to take the higher monthly payment due to the much lower total loan time spent at the adjustable rate. If they take the lower monthly payment, they will pay $928,560 in interest, which is $595,680 more than the $332,880 they would have paid if they had paid the extra $500 each month. This is because the loan spends a greater amount of time at a higher adjustable rate, which makes the loan much more expensive if you require a lower payment.

Relative to the graph of the approximated loan value with the fixed rate mortgage, the graph with variable interest after five years levels off slightly and then decreases at an increasing rate.

Relative to each other, the $4000 monthly payment curve takes longer to pay off and spends a longer time at a higher loan value, since the payment of $4000 struggles more to combat the increasing interest rate than the $4500 monthly payment. The $4500 payment curve also has a much steeper rate of decrease beyond the initial levelling off period, since the $4500 monthly payment is able to ‘attack' the remaining loan value much more aggressively.

4 Conclusion

For any loan, regardless of any personal circumstances, our friends should first consider the time in which they would like to move out.

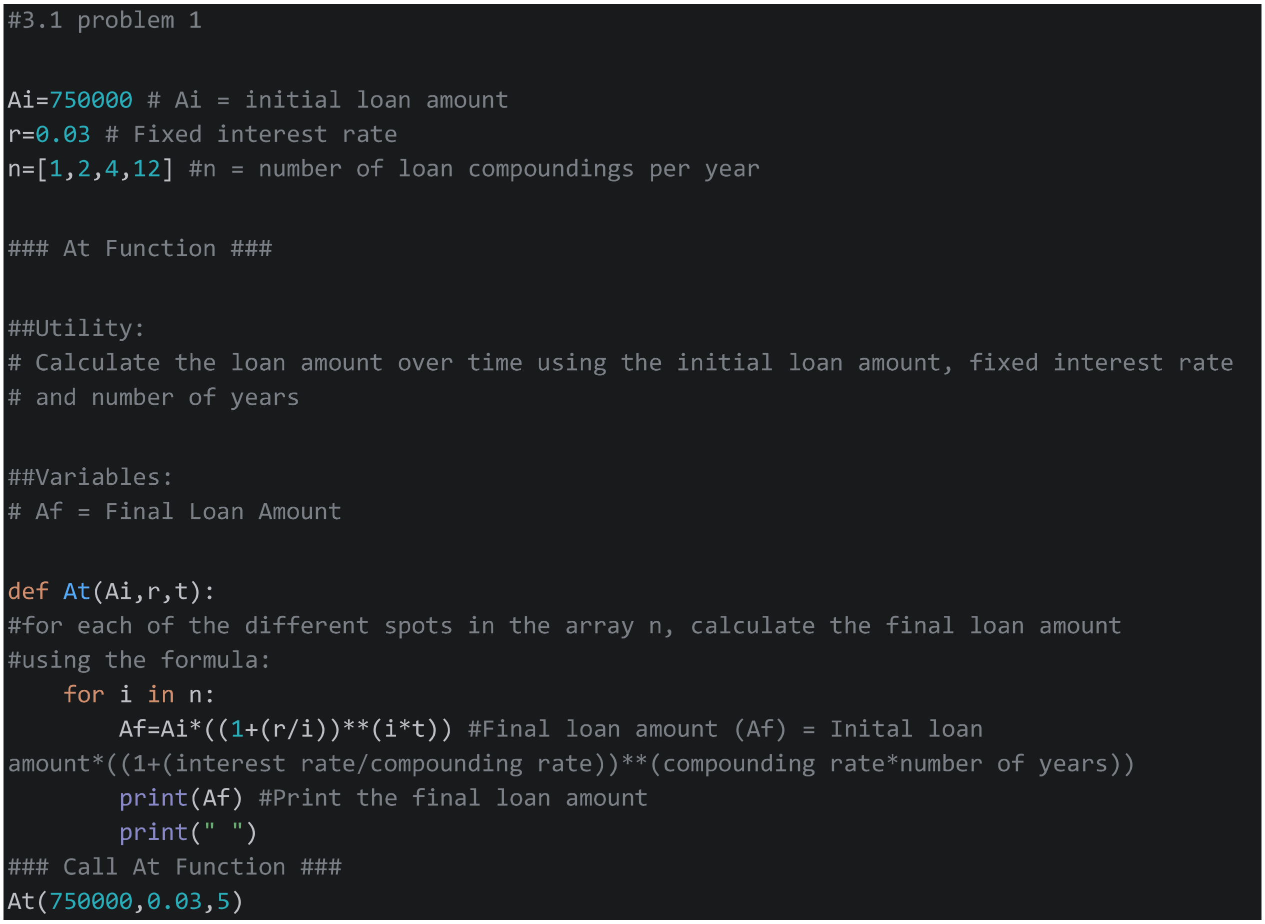
If our friends want to move out after a long period of time, we advise that they pursue a fixed-rate mortgage. If they anticipate having high enough income to cover a higher monthly payment, they should do so and pursue a shorter loan period. Even if they must pay a higher amount per month, they'll experience a lower interest rate over a shorter timeframe and save several hundreds of thousands of dollars.

If they don't predict sufficiently high income and have no choice but to take a longer loan period with a higher interest rate, we especially recommend that they put whatever savings they can towards a down payment. This will decrease the total amount they have to pay above the initial amount. This will benefit long-term loan takers even more than short-term loan takers.

If, however, our friends plan on moving out in a very short time, we recommend that they go for an adjustable-rate mortgage and aim to move out before the interest rate can begin increasing. This way they won't accrue much interest between when they move in and move out and will be able to select a low monthly payment, so their net loss will be very low by the time they move out. Relative to a short-term fixed interest rate loan that has the same interest rate as the first five years of the adjustable rate loan, the fixed rate loan will have a much higher monthly payment, so our friends will end up paying much more with a fixed rate loan than an adjustable rate loan. Thus, we recommend that our friends take an adjustable-rate loan for a quick move-out.

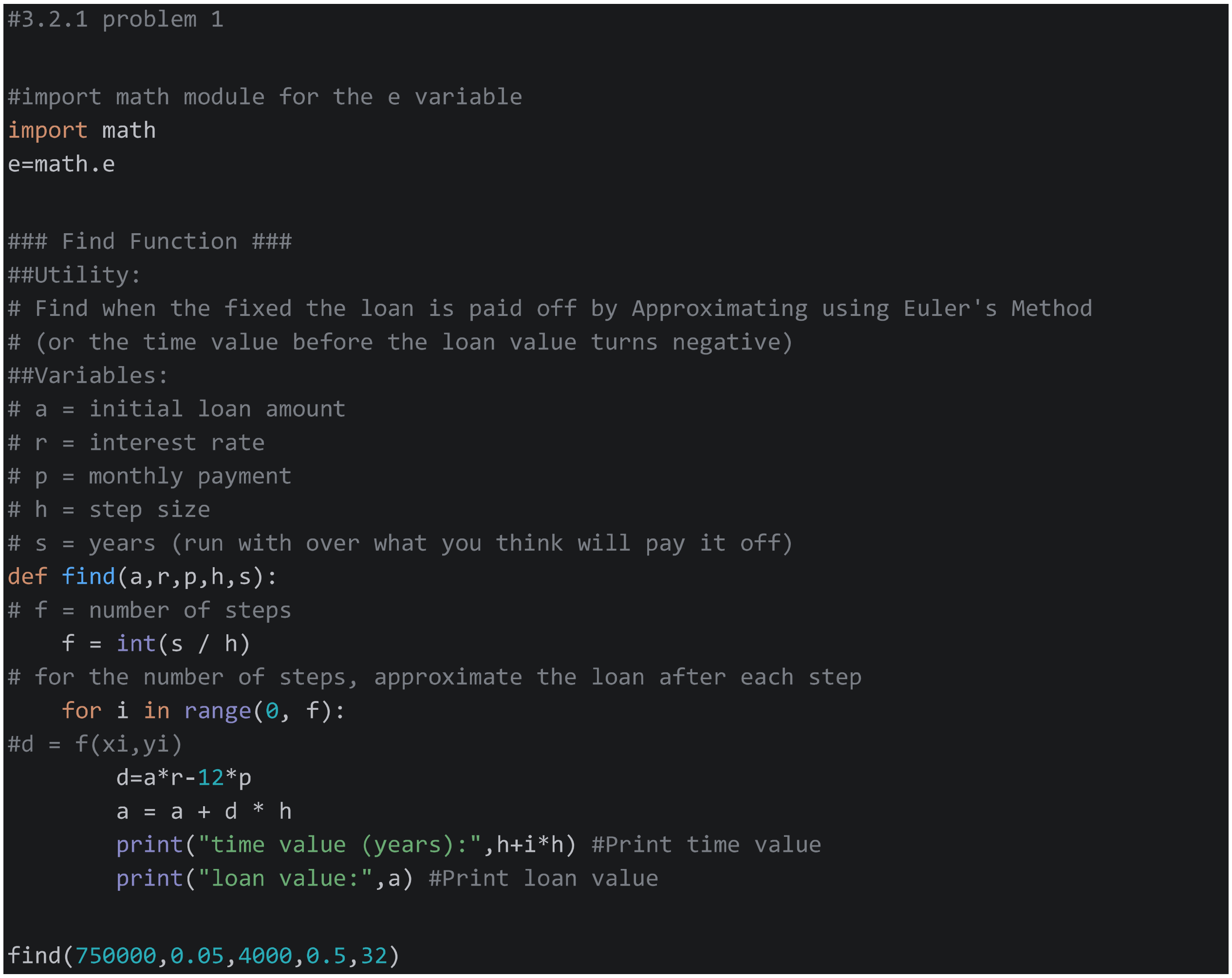
Appendix A: Python Code

3.1.1



```
#3.1 problem 1

Ai=750000 # Ai = initial loan amount
r=0.03 # Fixed interest rate
n=[1,2,4,12] #n = number of loan compoundings per year

### At Function ###

##Utility:
# Calculate the loan amount over time using the initial loan amount, fixed interest rate
# and number of years

##Variables:
# Af = Final Loan Amount

def At(Ai,r,t):
#for each of the different spots in the array n, calculate the final loan amount
#using the formula:
    for i in n:
        Af=Ai*((1+(r/i))**(i*t)) #Final loan amount (Af) = Inital loan
amount*((1+(interest rate/compounding rate))**(compounding rate*number of years))
        print(Af) #Print the final loan amount
        print(" ")
### Call At Function ###
At(750000,0.03,5)
```

3.2.1.1



```
#3.2.1 problem 1

#import math module for the e variable
import math
e=math.e

### Find Function ###
##Utility:
# Find when the fixed the loan is paid off by Approximating using Euler's Method
# (or the time value before the loan value turns negative)
##Variables:
# a = initial loan amount
# r = interest rate
# p = monthly payment
# h = step size
# s = years (run with over what you think will pay it off)
def find(a,r,p,h,s):
# f = number of steps
    f = int(s / h)
# for the number of steps, approximate the loan after each step
    for i in range(0, f):
#d = f(xi,yi)
        d=a*r-12*p
        a = a + d * h
        print("time value (years):",h+i*h) #Print time value
        print("loan value:",a) #Print loan value

find(750000,0.05,4000,0.5,32)
```

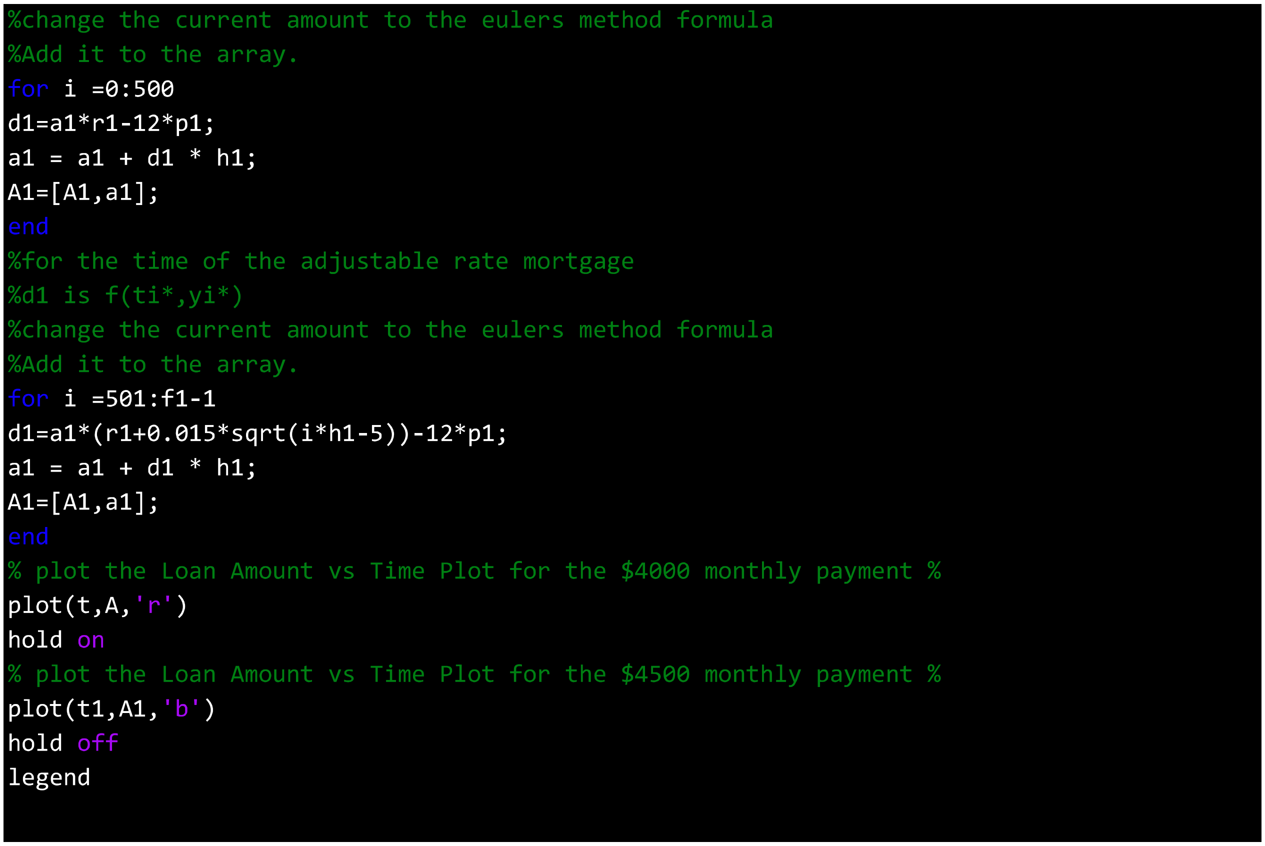
3.2.2.1

```
#3.2.2
#Import math for sqrt statement
import math

e=math.e
#a - loan amount
#r - interest rate
#p - monthly payment
#h - eulers method step size
#s - fixed rate years
#m - total years

### Find Function ###
def find(a,r,p,h,s,m):
#f = number of steps for the fixed loan
    f = int(s / h)
#Calculate the fixed loan over the number of steps
    for i in range(0, f+1):
        d=a*r-12*p
        a = a + d * h
        print("time value (years):",i*h+h)
        print("loan value:",a)
#g = number of steps for the adjustable loan
    g=int(m/h)
#Calculate the adjustable loan over the number of steps
    for i in range(int(f+1), g):
        d=a*(r+0.015*math.sqrt(i*h-5))-12*p
        a = a+d * h
#Print Statements
        print("time value (years):",i*h+h)
        print("loan value:",a)



### Call Find Function ###
find(750000,0.03,4000,0.01,5,35)
find(750000,0.03,4500,0.01,5,35)

#p=4000: at 34.97 yrs, a=0. --> 34.97*12*4000-750000 = $928560 in interest paid.
#p=4500: at 22.59 yrs, a=0. --> 22.56*12*4000-750000 = $332880 in interest paid.
```

Appendix B: Matlab Code

3.2.2.2

```
% For the $4000 monthly payment, plot the cost to time graph for the
% adjustable rate morgage
e=exp(1); %e variable
a=750000; %current amount variable initially set to initial loan amount
r=0.03; %fixed interest rate
p=4000; %monthly payment
h=0.01; %step size
s=35; %number of years
f = (s/h); %number of steps
t=0:0.01:35; %time interval (log the current Cost vs Time every 0.01 years for 35 years)
A=[750000]; %arrray containing the amounts A at times t.
%for the time of the fixed rate mortgage
%d1 is f(ti*,yi*)
%change the current amount to the eulers method formula
%Add it to the array.
for i =0:500
d=a*r-12*p;
a = a + d * h;
A=[A,a];
end
%for the time of the adjustable rate mortgage
%d1 is f(ti*,yi*)
%change the current amount to the eulers method formula
%Add it to the array.
for i =501:f-1
d=a*(r+0.015*sqrt(i*h-5))-12*p;
a = a + d * h;
A=[A,a];
end
% For the $4500 monthly payment and the data recieved from the python code
% plot the cost to time graph for the
% adjustable rate morgage
e=exp(1);
a1=750000; %current amount variable initially set to initial loan amount
r1=0.03; %fixed interest rate
p1=4500; %monthly payment
h1=0.01; %step size
s1=35; %number of years
f1 = (s1/h1); %number of steps
t1=0:0.01:35; %time interval (log the current Cost vs Time every 0.01 years for 35 years)
A1=[750000]; %arrray containing the amounts A at times t.
%for the time of the fixed rate mortgage
%d1 is f(ti*,yi*)
```

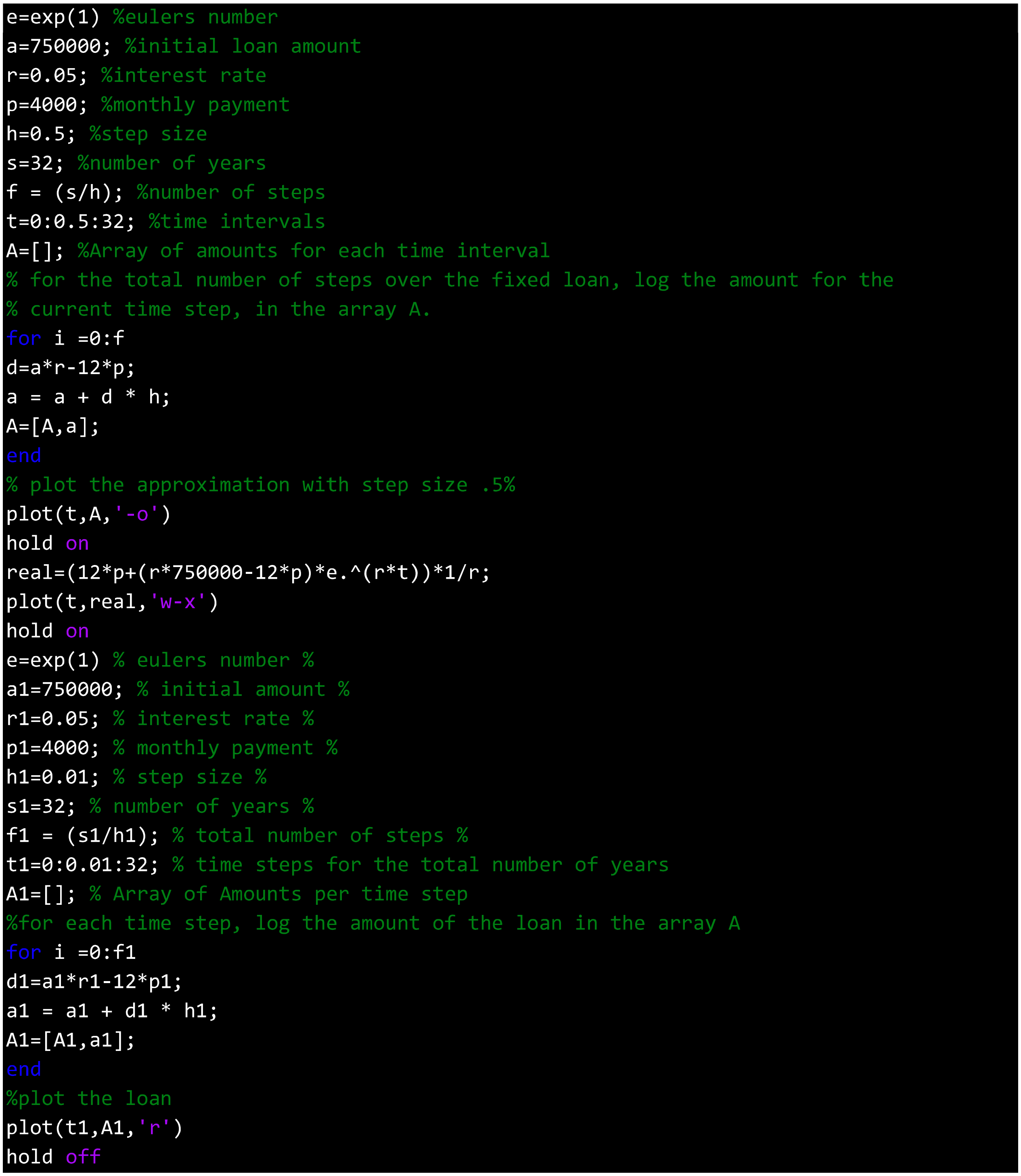


```
%change the current amount to the eulers method formula
%Add it to the array.
for i =0:500
d1=a1*r1-12*p1;
a1 = a1 + d1 * h1;
A1=[A1,a1];
end
%for the time of the adjustable rate mortgage
%d1 is f(ti*,yi*)
%change the current amount to the eulers method formula
%Add it to the array.
for i =501:f1-1
d1=a1*(r1+0.015*sqrt(i*h1-5))-12*p1;
a1 = a1 + d1 * h1;
A1=[A1,a1];
end
% plot the Loan Amount vs Time Plot for the $4000 monthly payment %
plot(t,A,'r')
hold on
% plot the Loan Amount vs Time Plot for the $4500 monthly payment %
plot(t1,A1,'b')
hold off
legend
```

3.1.1

```
e=exp(1); %e variable
Ai=750000; % initial loan amount
t=0:0.1:30; %time step interval
r=0.03; %interest rate
A4=Ai*((1+r/4).^(t*4)); %plot four compoundings a year at each time step
A12=Ai*((1+r/12).^(t*12)); %plot twelve compoundings a year at each time step
AC=Ai*e.^(r*t); %calculate continuous compoundings per year at each time step
%plot the graphs
plot(t,A4,'w')
hold on
plot(t,A12,'r')
hold on
plot(t,AC,'y')
hold off
legend
```

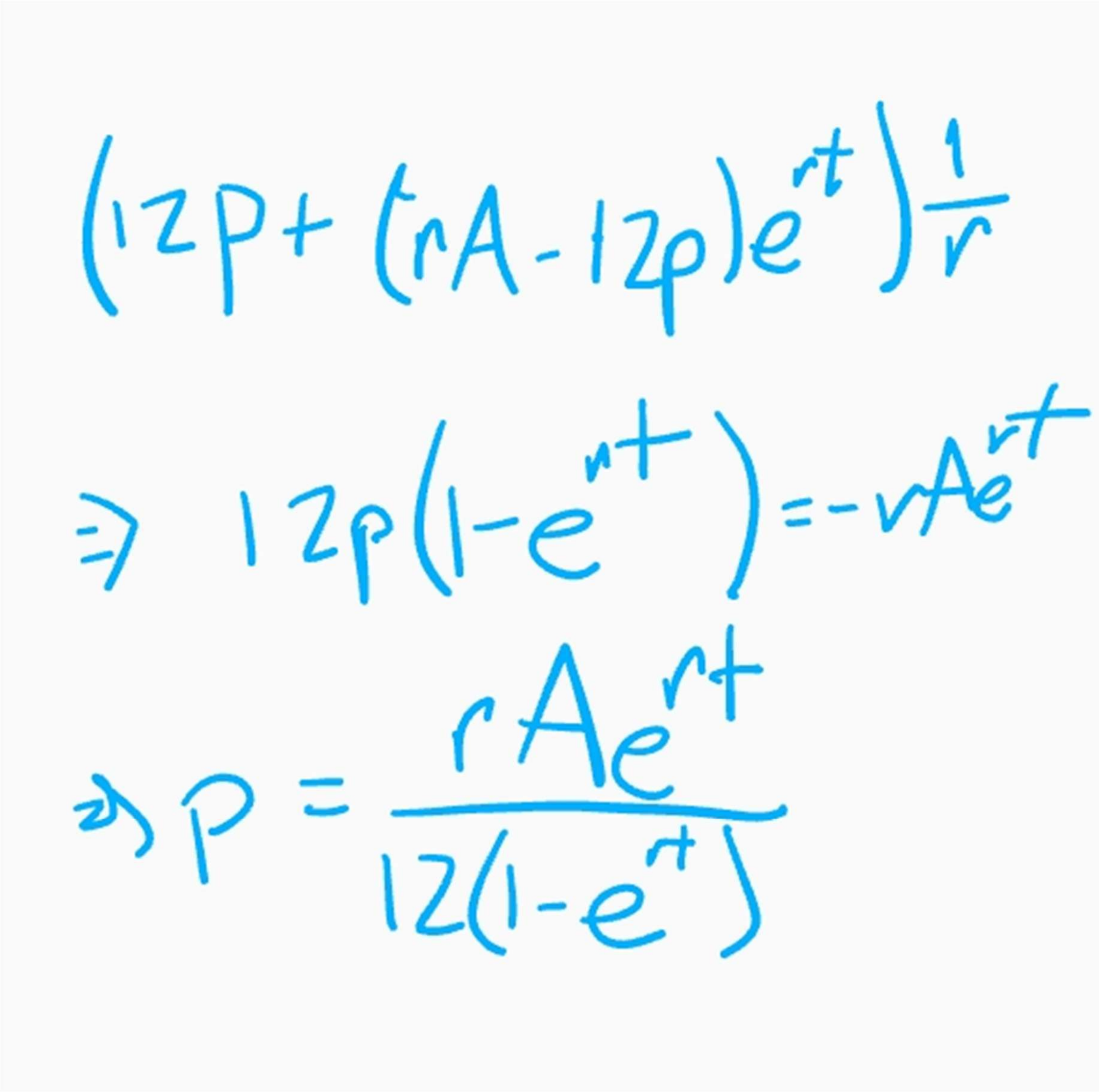
3.2.1.3



```
e=exp(1) %eulers number
a=750000; %initial loan amount
r=0.05; %interest rate
p=4000; %monthly payment
h=0.5; %step size
s=32; %number of years
f = (s/h); %number of steps
t=0:0.5:32; %time intervals
A=[]; %Array of amounts for each time interval
% for the total number of steps over the fixed loan, log the amount for the
% current time step, in the array A.
for i =0:f
d=a*r-12*p;
a = a + d * h;
A=[A,a];
end
% plot the approximation with step size .5%
plot(t,A,'-o')
hold on
real=(12*p+(r*750000-12*p)*e.^(r*t))*1/r;
plot(t,real,'w-x')
hold on
e=exp(1) % eulers number %
a1=750000; % initial amount %
r1=0.05; % interest rate %
p1=4000; % monthly payment %
h1=0.01; % step size %
s1=32; % number of years %
f1 = (s1/h1); % total number of steps %
t1=0:0.01:32; % time steps for the total number of years
A1=[]; % Array of Amounts per time step
%for each time step, log the amount of the loan in the array A
for i =0:f1
d1=a1*r1-12*p1;
a1 = a1 + d1 * h1;
A1=[A1,a1];
end
%plot the loan
plot(t1,A1,'r')
hold off
```

Appendix C: Hand Calculations

3.1.4



$$\left(12p + (rA - 12p)e^{rt}\right)\frac{1}{r}$$

$$\Rightarrow 12p\left(1 - e^{rt}\right) = -rAe^{rt}$$

$$\Rightarrow p = \frac{rAe^{rt}}{12(1 - e^{rt})}$$